

The Hitting Times of A Stochastic Epidemic Model

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Abstract

In this paper, we focus on the hitting times of a stochastic epidemic model presented by [9]. Under the help of the auxiliary stopping times, we investigate the asymptotic limits of the hitting times by the variations of calculus and the large deviation inequalities when the noise is sufficiently small. It can be shown that the relative position between the initial state and the hitting state determines the scope of the hitting times greatly.

1 Introduction

In [9], Gray, Greenhalgh, Hu, Mao and Pan discuss the asymptotic dynamics of a stochastic SIS epidemic model. Especially, they show the ergodic property and the recurrence of the model. Recently, there are also some other papers concerned on the ergodicity of stochastic epidemic models such as [10], [13] e.t.c. In these papers, to obtain the ergodicity and the recurrence, the noise is usually assumed to be small. According to the theory of Markov processes, the recurrence implies that it can reach any state in a finite time. Then another question arises: how long will it take? In this paper, we will investigate the asymptotic limits of the hitting times for any state for sufficiently small noise. This study may be helpful to the investigation of the rate under control of the disease transmission.

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Firstly, let us recall some notations and results in [9]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, and $\{B(t), t \geq 0\}$ be a scalar standard Brownian motion defined on the probability space. The stochastic version of the well known SIS model is given by the following Itô SDE

$$\begin{cases} dS(t) = [\mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t)]dt - \sigma S(t)I(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu + \gamma)I(t)]dt + \sigma S(t)I(t)dB(t). \end{cases}$$

Given that $S(t) + I(t) = N$, it is sufficient to study the following SDE for $I(t)$

$$dI(t) = I(t) ([\beta N - \mu - \gamma - \beta I(t)]dt + \sigma(N - I(t))dB(t)) \quad (1.1)$$

with initial value $I(0) = x \in (0, N)$. In [9], they showed that if $R_0^S := \frac{\beta N}{\mu + \gamma} - \frac{\sigma^2 N^2}{2(u+r)} > 1$, then the SDE (1.1) obeys

$$\limsup_{t \rightarrow \infty} I(t) \geq \xi, \quad \liminf_{t \rightarrow \infty} I(t) \leq \xi, \quad a.s.,$$

where $\xi = \sigma^{-2} \left(\sqrt{\beta^2 - 2\sigma^2(\mu + \gamma)} - (\beta - \sigma^2 N) \right)$ and $\lim_{\sigma \rightarrow 0} \xi = N - \frac{\mu + \gamma}{\beta}$ (Theorem 5.1 in [9]). This showed some recurrence of the model: $I(t)$ will rise to or above the level ξ infinitely often with probability one.

In fact, they actually showed the ergodic property and recurrence when $R_0^S > 1$ (Theorem 6.2 in [9]). That is to say, the SDE (1.1) can reach any point in $(0, N)$. According to some other papers concerned on the ergodicity of stochastic epidemic models such as [10], [13], the noise is usually assumed to be small enough to obtain the ergodicity. Therefore, in this paper, we are interested in the scopes of the hitting times when the noise is sufficiently small, i.e., how long will it take to arrive at any fixed point in $(0, N)$?

A question may arise: what about the other cases of σ when $R_0^S > 1$? Actually, the problem becomes much more complicated to solve and this paper is an attempt to investigate the limits of the hitting times for sufficiently small noise.

To emphasize the dependence of σ , we will denote the solution to (1.1) by $I^\sigma(\cdot)$ throughout this paper. Obviously, $I^0(\cdot)$ is the solution to the deterministic system. Now, we will formulate our question in the recurrent condition and assume that $R_0^D := \frac{\beta N}{\mu + \gamma} > 1$ for the sake of the recurrence throughout this paper (Obviously, which is equivalent to $R_0^S > 1$, if σ is sufficiently small). For any $y \in (0, N)$, define

$$\tau_y^\sigma := \inf\{t \geq 0; I^\sigma(t) = y\}.$$

Clearly, τ_y^σ is a stopping time, and we will investigate its asymptotic limit as $\sigma \rightarrow 0$. Obviously, by Theorem 3.1 in [9], $\tau_0^\sigma = \infty$, a.s., thus $\tau^\sigma = \tau_0^\sigma \wedge \tau_y^\sigma$, which is the exit time from $[0, y]$. Therefore, it is encouraged to consider the problem of exit from $[0, y]$. But the model (1.1) has a degenerate diffusion coefficient at 0, and starting from any neighborhood of the characteristic boundary 0, the hitting times of the other points in the neighborhood of 0 seem sufficiently large, which does not satisfy the conditions for the exit problem from a domain ([5]). Hence, we need to introduce the auxiliary stopping times, and investigate their asymptotic limits using the variations of calculus and the large deviation.

In this paper, we organize the sections as followed. In Section 2, we will introduce our main results. In Section 3, we will give some preliminaries used later. Section 4 will end this paper with the proof of main results.

2 Main results

Firstly, we will give some symbols. Define $*$ = $N - \frac{\mu+\gamma}{\beta}$, and

$$\bar{V}_y \triangleq \inf_{t>0} \inf_{u \in L^2([0,t])} \left\{ \frac{\int_0^t |u(s)|^2 ds}{2}; \phi(t) = y, \text{ where} \right. \\ \left. \phi(s) = * + \int_0^s \phi(\theta) [(\beta + \sigma u(\theta))(N - \phi(\theta)) - \mu - \gamma] d\theta \right\}.$$

Theorem 2.1. *For any $x, y \in (0, N)$, if $I^\sigma(0) = I^0(0) = x$, $T_y = \inf\{t \geq 0; I^0(t) = y\} = \frac{1}{\beta*} \ln \frac{y(x-*)}{x(y-*)}$, then for any $\delta > 0$,*

(1) *if $0 < x < *$, $* < y < N$ or $0 < y < x$, then*

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \left\{ e^{\frac{\bar{V}_y - \delta}{\sigma^2}} < \tau_y^\sigma < e^{\frac{\bar{V}_y + \delta}{\sigma^2}} \right\} = 1,$$

and $0 < \bar{V}_y < \infty$;

(2) *if $0 < x < *$, $x \leq y < *$, then*

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \{ |\tau_y^\sigma - T_y| > \delta \} = 1,$$

and $T_y < \infty$;

(3) *if $0 < x < *$ or $* < x < N$, $y = *$, then $\lim_{\sigma \rightarrow 0} \tau_*^\sigma = \infty$, and $\lim_{\sigma \rightarrow 0} \sigma^2 \ln \tau_*^\sigma = 0$ in probability;*

(4) *if $* < x < N$, $x < y < N$ or $0 < y < *$, then*

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \left\{ e^{\frac{\bar{V}_y - \delta}{\sigma^2}} < \tau_y^\sigma < e^{\frac{\bar{V}_y + \delta}{\sigma^2}} \right\} = 1,$$

and $0 < \overline{V}_y < \infty$;

(5) if $* < x < N$, $* < y < x$, then

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \{ |\tau_y^\sigma - T_y| > \delta \} = 1,$$

and $T_y < \infty$.

Remark 2.1. By the results of [9], we know that $\tau_y^\sigma < \infty$ a.s. for any $y \in (0, N)$. But the scopes of the hitting times depend on the relative position between the initial and the hitting states. Take $0 < x < *$ for an example. If $x < y < *$, then the hitting time τ_y^σ approaches a fixed constant with a large probability when the noise is small enough. But if $* < y < N$, the time to arrive at y is exponentially large about the noise σ with a large probability. This delicate description may help us understand the disease transmission better.

3 Preliminaries

Before the proofs of main results, we will give some well known results concerned on the problem of exit from a domain. The relevant literature may be found in [3], [4], [7], [8] etc and references therein. In this paper, we suggest [5] for reference.

Consider the SDE

$$\begin{cases} dx^\varepsilon(t) = b(x^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(x^\varepsilon(t))d\omega(t), \\ x^\varepsilon(t) \in \mathbb{R}^d, \quad x^\varepsilon(0) = x, \end{cases} \quad (3.1)$$

in the open, bounded domain $G \subset \mathbb{R}^d$, where $b(\cdot)$ and $\sigma(\cdot)$ are uniformly Lipschitz continuous functions of appropriate dimensions and $\omega(\cdot)$ is a standard Brownian motion.

Define the cost function

$$\begin{aligned} V(y, z, t) &\triangleq \inf \{ I_{y,t}(\phi); \phi \in C([0, t]) : \phi(t) = z \} \\ &= \inf \left\{ \frac{\int_0^t |u(s)|^2 ds}{2}; u \in L^2([0, t]), \phi(t) = z, \right. \\ &\quad \left. \text{where } \phi(s) = y + \int_0^s b(\phi(\theta))d\theta + \int_0^s \sigma(\phi(\theta))u(\theta)d\theta \right\}, \end{aligned}$$

where $I_{y,t}(\cdot)$ is the good rate function of (5.5.26) in [5], which controls the LDP (large deviation principles) associated with (3.1).

Define

$$V(y, z) \triangleq \inf_{t \geq 0} V(y, z, t).$$

Assumption (A-1) The unique stable equilibrium point in G of the d -dimensional ordinary differential equation

$$\dot{\phi}(t) = b(\phi(t)) \tag{3.2}$$

is at $0 \in G$, and

$$\phi(0) \in G \Rightarrow \forall t > 0, \phi(t) \in G \text{ and } \lim_{t \rightarrow \infty} \phi(t) = 0$$

Assumption (A-2) All the trajectories of the deterministic system (3.2) starting at $\phi(0) \in \partial G$ converge to 0 as $t \rightarrow \infty$.

Assumption (A-3) $\bar{V} \triangleq \inf_{z \in \partial G} V(0, z) < \infty$.

Assumption (A-4) There exists an $M < \infty$ such that, for all $\rho > 0$ small enough and all x, y with $|x - z| + |y - z| \leq \rho$ for some $z \in \partial G \cup \{0\}$, there is a function u satisfying that $\|u\| < M$ and $\phi(T(\rho)) = y$, where

$$\phi(t) = x + \int_0^t b(\phi(s))ds + \int_0^t \sigma(\phi(s))u(s)ds$$

and $T(\rho) \rightarrow 0$ as $\rho \rightarrow 0$.

Theorem 3.1. (Theorem 5.7.11 in [5]) Assume (A1)-(A4). For all $x \in G$ and all $\delta > 0$, $\tau^\varepsilon = \inf\{t \geq 0; x^\varepsilon(t) \in \partial G\}$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left\{ e^{\frac{\bar{V}-\delta}{\varepsilon}} < \tau^\varepsilon < e^{\frac{\bar{V}+\delta}{\varepsilon}} \right\} = 1.$$

Now, we turn to our proofs. We adapt the old symbols given above. In our case, $(0, N)$ play the same role as R^d in (3.1) and for any $y \in (0, N)$, $* = N - \frac{\mu+\gamma}{\beta}$ is the positive equilibrium of the deterministic model $I^0(\cdot)$ as 0 in (3.1). Note that $\bar{V}_y = V(*, y)$, and define $\bar{V}_\rho = V(*, \rho)$, $\bar{V}_{-\rho} = V(*, N - \rho)$ associated with the SDE (1.1) and $\tau_\rho^\sigma = \inf\{t \geq 0; I^\sigma(t) = \rho\}$, and $\tau_{-\rho}^\sigma = \inf\{t \geq 0; I^\sigma(t) = N - \rho\}$.

Lemma 3.1. For any positive sequence $\{T_n, n \geq 1\}$ such that $\sup_n T_n < \infty$ and sufficiently small $\rho_n > 0$, there exists a $M > 0$ such that

$$\limsup_{\sigma \rightarrow 0} \sigma^2 \log \mathbb{P}_{*-\delta_0} \left\{ \inf_{t \in [0, T_n]} I^\sigma(t) < 2\rho_n, I^\sigma(t) \in \left(\frac{\rho_n}{2}, * - \frac{\delta_0}{2}\right) \right\} \leq -\frac{(\ln \rho_n^{-1})^2}{8\sigma^2 T_n M}.$$

Proof. Note that

$$d \ln ((I^\sigma(t))^{-1}) = -d \ln I^\sigma(t) = \left\{ \beta I^\sigma(t) - \beta N + \mu + \gamma + \frac{\sigma^2(N - I^\sigma(t))^2}{2} \right\} dt - \sigma(N - I^\sigma(t))dB(t),$$

$$\ln(I^{-1}(0)) = \ln(* - \delta_0).$$

Therefore, there is $M > 0$ such that $|\ln((I^\sigma)^{-1}(t))| \leq MT_n + \sigma \sup_{t \in [0, T_n]} |\int_0^t (N - I^\sigma(s))dB(s)|$ for $t \in [0, T_n]$. Since $\rho_n > 0$ is sufficiently small, we may assume without loss of generality that $MT_n \leq \frac{\ln \rho_n^{-1}}{2}$, then

$$\begin{aligned} & \mathbb{P}_{*- \delta_0} \left\{ \inf_{t \in [0, T_n]} I^\sigma(t) < 2\rho_n, I^\sigma(t) \in \left(\frac{\rho_n}{2}, * - \frac{\delta_0}{2}\right) \right\} \\ & \leq \mathbb{P}_{*- \delta_0} \left\{ \sigma \sup_{t \in [0, T_n]} \left| \int_0^t (N - I^\sigma(s))dB(s) \right| > \frac{\ln \rho_n^{-1}}{2} \right\}. \end{aligned} \quad (3.3)$$

Let $M_n(t) = \exp \left\{ \lambda \sigma \int_0^t (N - I^\sigma(s))dB(s) - \frac{\lambda^2 \sigma^2}{2} \int_0^t (N - I^\sigma(s))^2 ds \right\}$, where $\lambda = \frac{(\ln \rho_n^{-1})}{2\sigma^2 T_n N^2} > 0$.

Then $\{M_n(t), t \geq 0\}$ is a sequence of martingale and

$$\begin{aligned} & \mathbb{P}_{*- \delta_0} \left\{ \sigma \sup_{t \in [0, T_n]} \left| \int_0^t (N - I^\sigma(s))dB(s) \right| > \frac{\ln \rho_n^{-1}}{2} \right\} \\ & \leq \mathbb{P}_{*- \delta_0} \left\{ \sigma \sup_{t \in [0, T_n]} M_n(t) > \exp \left\{ \frac{(\ln \rho_n^{-1})^2}{8\sigma^2 T_n M} \right\} \right\} \\ & \leq \exp \left\{ -\frac{(\ln \rho_n^{-1})^2}{8\sigma^2 T_n M} \right\}, \end{aligned} \quad (3.4)$$

where the last inequality is derived by the exponential martingale inequality. (3.3) and (3.4) implies the desired result. \square

Lemma 3.2. *For any sufficiently small $\delta_0 > 0$, let*

$$\overline{V}_{m, \rho} := \inf_{T \leq m} \inf_{\phi \in C([0, T]), \phi(T) = \rho} I_{*- \delta_0, T}(\phi).$$

If $\overline{V}_{m, \rho} < \infty$, then there is a decreasing $\phi(\cdot) \in C([0, T])$ for some $T \leq m$ such that $\phi(T) = \rho$ for the first time and

$$\overline{V}_m = I_{*- \delta_0, T}(\phi), \quad \phi(t) \in [\rho, * - \delta] \text{ for any } t \in [0, T].$$

Proof. Since $\{\phi \in C([0, T]), \phi(T) = \rho\}$ is the closed set of $C([0, T])$ and $I_{*- \delta_0, T}(\cdot)$ is a good rate function, there exists a $\phi_T \in C([0, T]), T \leq m$ such that

$$\inf_{\phi \in C([0, T]), \phi(T) = \rho} I_{*- \delta_0, T}(\phi) = I_{*- \delta_0, T}(\phi_T).$$

Therefore, there is a sequence of $\{T_n, n \geq 1\}$ and $\{\phi_n, n \geq 1\}$ such that $T_n \leq m$, $\phi_n \in C([0, T_n])$ and

$$\phi_n(T_n) = \rho, \quad I_{*-\delta_0, T_n}(\phi_n) \rightarrow \overline{V}_{m, \rho}.$$

Define

$$\tau_n = \inf\{t \geq 0; \phi_n(t) = \rho\}.$$

Then $\tau_n \leq T_n$ and consider $\{\phi_n(t), t \in [0, \tau_n]\}$. Since $\phi_n(\tau_n) = \rho$, $\overline{V}_m \leq I_{*-\delta, \tau_n}(\phi_n) \leq I_{*-\delta_0, T_n}(\phi_n)$. Therefore, we may assume that $\phi_n(T_n) = \rho$ for the first time without loss of generality.

Similarly, define

$$\tilde{\tau}_n = \sup\{0 \leq t \leq T_n; \phi_n(t) = * - \delta\}.$$

Consider $\{\phi_n(t), t \in [\tilde{\tau}_n, T_n]\}$. Since $\phi_n(\tilde{\tau}_n) = * - \delta$ and $\phi_n(T_n) = \rho$, we may construct by homogeneity a trajectory $\{\tilde{\phi}_n(t), t \in [0, T_n - \tilde{\tau}_n]\}$ such that $\tilde{\phi}_n(0) = * - \delta$, $\tilde{\phi}_n(T_n - \tilde{\tau}_n) = \rho$ and $\overline{V}_{m, \rho} \leq I_{*-\delta_0, T_n - \tilde{\tau}_n}(\tilde{\phi}_n) \leq I_{*-\delta_0, T_n}(\phi_n)$. Therefore, we may also assume that $\phi_n(t) \leq * - \delta$ for any $t \in [0, T_n]$ without loss of generality.

In all, there is a sequence of $\{T_n, n \geq 1\}$ and $\{\phi_n, n \geq 1\}$ such that $T_n \leq m$, $\phi_n \in C([0, T_n])$, $\phi_n(T_n) = \rho$, $I_{*-\delta_0, T_n}(\phi_n) \rightarrow \overline{V}_m$, and $\phi_n(t) \in [\rho, * - \delta]$ for $t \in [0, T_n]$.

Note that $T_n \leq m$, we may assume that $T_n \uparrow T \leq m$ without loss of generality. For $t \in [T_n, T]$, let $u_n(t) = 0$ and $\phi_n(t) = \rho + \int_{T_n}^t \phi_n(s)(N - \mu - \gamma - \beta\phi_n(s))ds$. Then $\phi_n(T) \rightarrow \rho$ as $n \rightarrow \infty$, $I_{*-\delta, T}(\phi_n) = I_{*-\delta, T_n}(\phi_n)$ and $\phi_n(t) \in [\rho, * - \delta]$ for $t \in [0, T]$ if ρ is sufficiently small.

Since $\overline{V}_{m, \rho} < \infty$, $I_{*-\delta_0, T}(\phi_n) = \frac{\int_0^T |u_n(t)|^2 dt}{2} \leq \overline{V}_{m, \rho} + 1$, we may assume that $\{\phi_n, n \geq 1\}$ converges to ϕ in $C([0, T])$. Alike the proof of Lemma 1.4.17 in [6], we could show that $\phi_n \rightarrow \phi$ in $C([0, T])$ and ϕ_n converges weakly to ϕ in H_T^1 . Therefore, u_n converges weakly to u in $L^2([0, T])$, where

$$\phi_t = * - \delta_0 + \int_0^t \phi(s)(N - \mu\gamma - \beta\phi(s))ds + \int_0^t \phi(s)(N - \phi(s))u(s)ds. \quad (3.5)$$

By Banach-Steinhaus Theorem,

$$I_{*-\delta, T}(\phi) \leq \liminf_{n \rightarrow \infty} I_{*-\delta, T}(\phi_n) = \overline{V}_m.$$

Obviously, $I_{*-\delta, T}(\phi) \geq \overline{V}_m$. Therefore, $\overline{V}_m = I_{*-\delta, T}(\phi)$, where $T \leq m$, $\phi(T) = \rho$ for the first time and $\phi(t) \in [\rho, * - \delta]$ for any $t \in [0, T]$.

In fact, we may assume that $\phi(\cdot)$ is nonincreasing in $[0, T]$. Otherwise, there are $0 \leq t_1 < t_2 \leq T$ such that $\phi(t_1) < \phi(t_2)$. Since $\phi(\cdot)$ is continuous and $\phi(T) = \rho$, there exists a $t_3 > t_2$ such that

$\phi(t_3) = \phi(t_1)$. If $u(t) \equiv 0$ a.s. for $t \in [t_1, t_3]$, then $\phi \in [\rho, * - \delta]$ and $\phi(t) = \phi(t_1) + \int_{t_1}^t \phi(s)(N - \mu - \gamma - \beta\phi(s))ds$ is increasing in $[t_1, t_3]$. This contradicts the assumption $\phi(t_2) > \phi(t_3)$. This means that $u \neq 0$ a.s. in $[t_1, t_3]$. We could omit the time between t_1 and t_3 , and splice the trajectory in $[0, t_1]$ with the trajectory in $[t_3, T]$, and get a new trajectory $\tilde{\phi}(\cdot)$ in $C([0, T - (t_3 - t_1)])$ such that $\tilde{\phi}(0) = * - \delta$ and $\tilde{\phi}(T - (t_3 - t_1)) = \rho$ which satisfies

$$d\tilde{\phi}(t) = \tilde{\phi}(t)(\beta N - \mu - \gamma - \beta\tilde{\phi}(t))dt + \tilde{\phi}(t)(N - \tilde{\phi}(t))\tilde{u}(t)dt,$$

in $[0, T - (t_3 - t_1)]$, where $\tilde{u} \in L^2([0, T - (t_3 - t_1)])$ is defined according to u by splice. Since $u \neq 0$ a.s. in $[t_1, t_3]$, thus $\frac{\int_0^{T-(t_3-t_1)} |\tilde{u}(s)|^2 ds}{2} < \frac{\int_0^T |u(s)|^2}{2} = \bar{V}_{m,\rho}$, which contradicts the definition of $\bar{V}_{m,\rho}$. Therefore, we may assume that $\phi(\cdot)$ is decreasing in $[0, T]$. □

Proposition 3.1. *For any $0 < \rho < y$,*

$$\lim_{\rho \rightarrow 0} \bar{V}_\rho = \lim_{\rho \rightarrow 0} \bar{V}_{-\rho} = \infty.$$

Proof. We will give the proof of V_ρ , and the same method holds for $\bar{V}_{-\rho}$.

Note that \bar{V}_ρ is nondecreasing as $\rho \rightarrow 0$. Therefore, if

$$\bar{V} := \lim_{\rho \rightarrow 0} \bar{V}_\rho < \infty, \tag{3.6}$$

then for sufficiently small $\rho > 0$, we have $\bar{V}_\rho \leq \bar{V} < \infty$.

Let $u(t) \equiv -\frac{2r_0\beta^2}{2r_0\beta + \mu + \gamma}$, where r_0 is a fixed constant such that $0 < 2r_0 < *$, and

$$\begin{aligned} \phi_t &= * + \int_0^t \phi(s)(N - \mu\gamma - \beta\phi(s))ds + \int_0^t \phi(s)(N - \phi(s))u(s)ds \\ &= * + \int_0^t \frac{\beta(\mu + \gamma)}{2r_0\beta + \mu + \gamma} \phi(s)(* - 2r_0 - \phi(s))dt. \end{aligned}$$

Then there exist positive and sufficiently small δ_0 and t_0 such that $\phi(t_0) = * - \delta_0$ and $\frac{\int_0^{t_0} |u(s)|^2 ds}{2} < \frac{\bar{V}}{2}$, which implies that

$$V(*, * - \delta_0) < \frac{\bar{V}}{2}. \tag{3.7}$$

Note that $V(*, \rho) \geq V(*, * - \delta_0) + V(* - \delta_0, \rho)$, thus (3.6) and (3.7) implies that

$$\lim_{\rho \rightarrow 0} V(* - \delta_0, \rho) \leq \frac{\bar{V}}{2} < \infty. \tag{3.8}$$

Note that

$$V(* - \delta_0, \rho) = \inf_{m>0} \bar{V}_{m,\rho},$$

thus Lemma 3.2 and (3.8) implies that there exists a sequence of $\rho_n \rightarrow 0$ as $n \rightarrow \infty$, and there is a $I_{*- \delta_0, T_n}(\phi_n)$ such that $\phi_n(\cdot)$ is decreasing, contained in $[\rho_n, * - \delta_0]$ and

$$\sup_n I_{*- \delta_0, T_n}(\phi_n) = \sup_n \frac{\int_0^{T_n} |u_n(t)|^2 dt}{2} < \infty, \quad (3.9)$$

where the relationship between ϕ_n and u_n is defined as (3.5).

By Lemma 3.2, $\phi'_n(t) \leq 0$ and $\phi_n(t) \in [\rho_n, * - \delta_0]$ for $t \in [0, T_n]$. Then

$$\phi_n(t) \left(\tilde{\beta}N - \mu - \gamma - \tilde{\beta}\phi(t) \right) \leq 0,$$

where $\tilde{\beta} = \beta + u_n(t)$.

By computation, for any $t \in [0, T_n]$,

$$\begin{aligned} u_n(t) &\leq \frac{\mu + \gamma}{1 - \phi(t)} - \beta \\ &= \frac{\beta(\phi(t) - *)}{N - \phi(t)} \leq -\frac{\beta\delta_0}{N - *}. \end{aligned}$$

Therefore, $|u_n(t)| \geq \frac{\beta\delta_0}{N - *}$ for $t \in [0, T_n]$. So $I_{*- \delta_0, T_n}(\phi_n) \geq \frac{\beta^2\delta_0^2 T_n}{2(N - *)^2}$.

By (3.9), $\sup_{n \rightarrow \infty} T_n < \infty$. Similarly, we may also assume that for some $M > 0$

$$\sup_n I_{*- \delta_0, T_n}(\phi_n) \leq M. \quad (3.10)$$

By the discussion above, $\phi_n \in [\rho_n, * - \delta_0]$ for $t \in [0, T_n]$, $\sup_{n \rightarrow \infty} T_n < \infty$ and $\sup_n I_{*- \delta_0, T_n}(\phi_n) \leq M$.

On the other hand, by the lower bound of the large deviation principle,

$$\begin{aligned} \liminf_{\sigma \rightarrow 0} \sigma^2 \log \mathbb{P}_{*- \delta_0} \left\{ \inf_{t \in [0, T_n]} I^\sigma(t) < 2\rho_n, I^\sigma(t) \in \left(\frac{\rho_n}{2}, * - \frac{\delta_0}{2}\right) \right\} \\ \geq -\inf \left\{ I_{*- \delta_0, T_n}(\phi); \inf_{t \in [0, T_n]} \phi < 2\rho_n, \phi \in \left(\frac{\rho_n}{2}, * - \frac{\delta_0}{2}\right) \right\}. \end{aligned}$$

Then by Lemma 3.1, for any $\inf_{t \in [0, T_n]} \phi < 2\rho_n, \phi \in \left(\frac{\rho_n}{2}, * - \frac{\delta_0}{2}\right)$,

$$I_{*- \delta_0, T_n}(\phi) \geq \frac{(\ln \rho_n^{-1})^2}{8\sigma^2 T_n M}.$$

Especially, $\{\phi_n\}$ satisfies the above conditions and then $I_{*- \delta_0, T_n}(\phi_n) \geq \frac{(\ln \rho_n^{-1})^2}{8\sigma^2 T_n M}$.

Since $\sup_n T_n < \infty$ and $\lim_{n \rightarrow \infty} \rho_n = 0$, we have

$$\lim_{n \rightarrow \infty} I_{*-\delta_0, T_n}(\phi_n) = \infty,$$

which contradicts (3.10). Therefore, The proof is completed. \square

Remark 3.1. By the definition of \bar{V}_ρ , we may prove that $\bar{V}_0 = \lim_{\rho \rightarrow 0} \bar{V}_\rho$. Therefore, what we have to do is just to prove that $\bar{V}_0 = \infty$. Note that $\bar{V}_0 < \infty$ is equivalent to the existence of $0 < T < \infty$ and $u(\cdot) \in L^2([0, T])$ such that $\phi(T) = 0$ and for $t \in [0, T]$,

$$\phi_t = * + \int_0^t \phi(s) (N - \mu\gamma - \beta\phi(s)) ds + \int_0^t \phi(s) (N - \phi(s)) u(s) ds.$$

Therefore, one may be initialized to investigate the positivity for the density of $I^\sigma(T)$ at 0 with the initial condition $I^\sigma(0) = *$ (see [1], [2], [12] and references therein). But it should be careful that the diffusion coefficient is degenerate at 0 and a simple computation implies that the Hörmander condition (we refer [11] and [12] for reference) are not satisfied. Therefore, the support theorems can not be applied directly. Here, we adapt the analysis of variation and the large deviation principle to get the desired results.

4 Proof of Main results

Proof of Theorem 2.1 (1) Firstly, note that

$$\begin{aligned} & \mathbb{P} \left\{ \tau_y^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_{\rho+\delta}}{\sigma^2}} \right\} \\ & \leq \mathbb{P} \left\{ \tau_y^\sigma \wedge \tau_\rho^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_{\rho+\delta}}{\sigma^2}} \right\} + \mathbb{P} \left\{ \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_{\rho+\delta}}{\sigma^2}} \right\} \\ & \leq \mathbb{P} \left\{ \tau_y^\sigma \wedge \tau_\rho^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_{\rho+\delta}}{\sigma^2}} \right\} + \mathbb{P} \left\{ e^{\frac{\bar{V}_\rho \wedge \bar{V}_{-\rho-\delta}}{\sigma^2}} \leq \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_{\rho+\delta}}{\sigma^2}} \right\} \\ & + \mathbb{P} \left\{ \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma < e^{\frac{\bar{V}_\rho \wedge \bar{V}_{-\rho-\delta}}{\sigma^2}} \right\} \\ & := P_{\sigma, \rho}^1 + P_{\sigma, \rho}^2 + P_{\sigma, \rho}^3. \end{aligned}$$

In the following paragraph, we will give their estimation respectively.

In the model of (1.1), the equilibrium of the deterministic system is $*$ and consider the boundary $\partial G = \{\rho, y\}$ for $y > *$ and ρ is sufficiently small. It can be verified that the Assumptions (A-1)–(A-4) are satisfied. We will give the detail of them below.

The Assumption (A-1) and (A-2) are easily verified. For (A-3), let $u_t = u$ sufficiently large such that $N - \frac{\mu+\gamma}{\beta+u} > y$, then for the deterministic system

$$\phi(t) = * + \int_0^t \phi(s)(\beta N - \mu - \gamma - \beta\phi(s))ds + \int_0^t \phi(s)(N - \phi(s))u(s)ds, \quad (4.1)$$

there exists a $T > 0$ such that $\phi(T) = y$ and $\bar{V}_y \leq \frac{\int_0^T u^2(s)ds}{2} < \infty$. Meanwhile, for any x_1, x_2 sufficiently close to each other in the neighborhood of y , there exists $T(\rho)$ such that $\phi(T(\rho)) = x_2$, (4.1) holds and $T(\rho) \rightarrow 0$ as $\rho \rightarrow 0$. When x_1, x_2 are sufficiently close to each other in the neighborhood of ρ or $*$, we can get the same results. Then Assumption (A-4) holds.

Therefore, for any $y \in (*, N)$, Theorem 3.1 implies

$$\lim_{\sigma \rightarrow 0} P_{\sigma, \rho}^1 = \lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_y^\sigma \wedge \tau_\rho^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_\rho + \delta}{\sigma^2}} \right\} = 0.$$

Similarly,

$$\lim_{\sigma \rightarrow 0} P_{\sigma, \rho}^3 = 0.$$

What is left is the estimation of $P_{\sigma, \rho}^2$. In fact, by Proposition 3.1,

$$\lim_{\rho \rightarrow 0} \bar{V}_\rho = \lim_{\rho \rightarrow 0} \bar{V}_{-\rho} = \infty.$$

Thus let ρ be sufficiently small such that $\bar{V}_\rho \wedge \bar{V}_{-\rho} > \bar{V}_y + 2\delta$, which implies

$$\lim_{\sigma \rightarrow 0} P_{\sigma, \rho}^2 = 0.$$

Therefore, $\lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_y^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_\rho + \delta}{\sigma^2}} \right\} = 0$.

Since $\bar{V}_y \wedge \bar{V}_\rho = \bar{V}_y$ for sufficiently small ρ ,

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_y^\sigma > e^{\frac{\bar{V}_y + \delta}{\sigma^2}} \right\} = \lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_y^\sigma > e^{\frac{\bar{V}_y \wedge \bar{V}_\rho + \delta}{\sigma^2}} \right\} = 0.$$

The proof of upper bound ends.

Now, we turn to the proof of the lower bound.

$$\begin{aligned} & \mathbb{P} \left\{ \tau_y^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_\rho - \delta}{\sigma^2}} \right\} \\ & \leq \mathbb{P} \left\{ \tau_y^\sigma \wedge \tau_\rho^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_\rho - \delta}{\sigma^2}} \right\} + \mathbb{P} \left\{ \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_\rho - \delta}{\sigma^2}} \right\} \\ & \leq \mathbb{P} \left\{ \tau_y^\sigma \wedge \tau_\rho^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_\rho - \delta}{\sigma^2}} \right\} + \mathbb{P} \left\{ e^{\frac{\bar{V}_\rho \wedge \bar{V}_{-\rho} - \delta}{\sigma^2}} \leq \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma \leq e^{\frac{\bar{V}_y \wedge \bar{V}_\rho \wedge \bar{V}_{-\rho} - \delta}{\sigma^2}} \right\} \\ & + \mathbb{P} \left\{ \tau_\rho^\sigma \wedge \tau_{-\rho}^\sigma < e^{\frac{\bar{V}_\rho \wedge \bar{V}_{-\rho} - \delta}{\sigma^2}} \right\} \\ & := Q_{\sigma, \rho}^1 + Q_{\sigma, \rho}^2 + Q_{\sigma, \rho}^3. \end{aligned}$$

The lower bound can be proved in the same way.

Now, we turn to the proof of $0 < \bar{V}(y) < \infty$. Since $0 < y < N$, let $u(t) \equiv u$ be sufficiently large such that $N - \frac{\mu+\gamma}{\beta+u} > y$, then by (1.2) in [9], there exists $T > 0$ such that $\phi(T) = y$, where $\phi(t) = x + \int_0^t \phi(s)(\beta N - \mu - \gamma - \beta\phi(s))ds + \int_0^t \phi(s)(N - \phi(s))u(s)ds$. Thus, by the definition of \bar{V}_y , $\bar{V}_y < \frac{u^2 T}{2} < \infty$.

Let $* < \delta < y$, note that $\bar{V}_y \geq V(y - \delta, y)$. Then $\bar{V}_y = 0$ implies $V(y - \delta, y) = 0$. Then there are two sequences of $\{T_n, n \geq 1\}$ and $\{\phi_n, n \geq 1\}$ such that $I_{y-\delta, T_n}(\phi_n) \rightarrow 0$, where $\phi_n(0) = y - \delta$, $\phi_n(T_n) = y$, $\frac{\int_0^{T_n} |u_n(t)|^2}{2} = I_{y-\delta, T_n}(\phi_n) \rightarrow 0$, and

$$\phi_n(t) = y - \delta + \int_0^t \phi_n(s)(\beta N - \mu - \gamma - \beta\phi_n(s))ds + \int_0^t \phi_n(s)(N - \phi_n(s))u_n(s)ds$$

for all $t \in [0, T_n]$. Alike the proof of Proposition 3.1, we could show that $T_n \rightarrow 0$. It is easy to prove that $\phi_n(T_n)$ converges to $y - \delta$. This contradicts the fact that $\phi_n(T_n) = y$. Therefore, $\bar{V}_y > 0$ for $* < y < N$.

(2) For any $0 < y < *$, let $T_y = \inf \{t \geq 0; I^0(t) = y\}$, then $T_y < \infty$, and for any $\delta > 0$, we also define $d(\delta) := \min \{I^0(T_y + \delta) - y, y - I^0(T_y - \delta)\} > 0$ accordingly. Since the coefficients of $I^\sigma(\cdot)$ and $I^0(\cdot)$ are of uniformly bounded Lipschitz, there exists $M > 0$ such that

$$G_t^\sigma \leq M \int_0^t G_s^\sigma ds + \sigma \sup_{s \leq t} \int_0^s |I^\sigma(s)(N - I^\sigma(s))dB_s|,$$

where $G_t^\sigma = \sup_{s \leq t} |I^\delta(s) - I^0(s)|$. Then by Gronwall's inequality, there exists $M' > 0$ such that

$$G_{T_y+\delta}^\sigma \leq M' \sigma \sup_{s \leq T_y+\delta} \int_0^s |I^\sigma(s)(N - I^\sigma(s))dB_s|, \text{ a.s.}$$

Therefore, there are $M'' > 0$ and $M''' > 0$ such that

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in [0, T_y+\delta]} |I^\sigma(t) - I^0(t)| > \frac{d(\delta)}{2} \right\} &\leq \mathbb{P} \left\{ \sigma \sup_{s \leq T_y+\delta} \int_0^s |I^\sigma(s)(N - I^\sigma(s))dB_s| > M'' d(\delta) \right\} \\ &\leq M''' \sigma^2 E \int_0^{T_y+\delta} |I^\sigma(s)(N - I^\sigma(s))|^2 ds, \end{aligned}$$

where the last inequality is derived by the B-D-G inequality for continuous martingales.

Therefore,

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \sup_{t \in [0, T_y+\delta]} |I^\sigma(t) - I^0(t)| > \frac{d(\delta)}{2} \right\} = 0.$$

By the definition of $d(\delta)$ and T_y , $\sup_{t \in [0, T_y+\delta]} |I^\sigma(t) - I^0(t)| \leq \frac{d(\delta)}{2}$ implies that $T_y - \delta < \tau_y^\sigma < T_y + \delta$.

Hence,

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \{T_y - \delta < \tau_y^\sigma < T_y + \delta\} = 1.$$

(3) Firstly, note that T_y is increasing, then $\lim_{y \rightarrow *} T_y = \sup_{y < *} T_y =: T_0$. If $T_0 < \infty$, by the definition of T_y and the continuity of $I^0(\cdot)$, $I^0(T_0) = *$, which contradicts the trajectory property of $I^0(\cdot)$. Therefore, $\lim_{y \rightarrow *} T_y = \infty$.

Since for any $y < *$, $\tau_y^\sigma < \tau_*^\sigma$ and

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \{T_y - \delta < \tau_y^\sigma < T_y + \delta\} = 1,$$

for any $M > 0$, we have

$$\lim_{\sigma \rightarrow 0} \mathbb{P}_x \{\tau_y^\sigma > M\} = 1, \quad (4.2)$$

i.e., $\lim_{\sigma \rightarrow 0} \tau_y^\sigma = \infty$ in probability.

Next, We will show that $\lim_{y \downarrow *} \bar{V}_y = 0$. Let $u_t = u$ sufficiently large such that $N - \frac{\mu + \gamma}{\beta + u} > *$, then there exists a trajectory

$$\phi(t) = * + \int_0^t \phi(s) [(\beta + u)N - \mu - \gamma - (\beta + u)\phi(s)] dt$$

and $\phi(T_y) = y$ such that $T_y \rightarrow 0$ as $y \rightarrow *$. Therefore, $\lim_{y \downarrow *} \bar{V}_y = 0$.

For any $\delta > 0$, let y be sufficiently close to y^* and $y > y^*$ such that $\bar{V}_y \leq \frac{\delta}{2}$, then

$$\limsup_{\sigma \rightarrow 0} \mathbb{P} \{\sigma^2 \log \tau_*^\sigma > \delta\} \leq \limsup_{\sigma \rightarrow 0} \mathbb{P} \left\{ \sigma^2 \log \tau_y^\sigma - \bar{V}_y > \frac{\delta}{2} \right\} = 0, \quad (4.3)$$

where the last inequality is derived by (1) in Theorem 2.1. Therefore, $\limsup_{\sigma \rightarrow 0} \sigma^2 \log \tau_*^\sigma \leq 0$.

Let $x < y < *$, then $\tau_*^\sigma > \tau_y^\sigma$ and $\lim_{\sigma \rightarrow 0} \tau_y^\sigma = T_y$, where $0 < T_y < \infty$. Therefore,

$$\begin{aligned} & \limsup_{\sigma \rightarrow 0} \mathbb{P} \{\sigma^2 \log \tau_*^\sigma < -\delta\} \\ & \leq \limsup_{\sigma \rightarrow 0} \mathbb{P} \left\{ \sigma^2 \log \tau_y^\sigma < -\delta, \tau_y^\sigma \geq \frac{T_y}{2} \right\} + \limsup_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_y^\sigma < \frac{T_y}{2} \right\} = 0. \end{aligned} \quad (4.4)$$

where the last inequality is derived by (4.2) (2) in Theorem 2.1.

Therefore, for any $\delta > 0$, (4.3) and (4.4) implies $\limsup_{\sigma \rightarrow 0} \mathbb{P} \{|\sigma^2 \log \tau_*^\sigma| > \delta\} = 0$.

The rest proof of (4)-(6) is similar to (1)-(3), so we omit it. Thus, the proof is completed.

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